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THE CONJUGACY VECTOR OF A p-GROUP OF MAXIMAL CLASS*

BY

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ABSTRACT

We find the conjugacy vector, i.e., we determine the number of conjugacy classes which compose the sets of the elements with centralizers of equal order, for several general families of p-groups of maximal class which include those of order up to p^9. As a consequence, we obtain the number of conjugacy classes, $r(G)$, for the groups in these families. Also, we provide upper and lower bounds for $r(G)$ and characterize when they are attained. Examples are given showing that the bounds are actually attained.

Introduction

It is well-known that a p-group G has maximal class if and only if there exist elements with centralizer of order p^2 (see [3], p. 375). Moreover, all such elements form $(p-1)^2$ or p^2-p conjugacy classes, according as G has degree of commutativity zero or not. In this paper, we pose the more general problem of finding *all* the orders of the centralizers of elements in G and determining the number of conjugacy classes which make up each set of elements with centralizers of the same order. In the cases we have succeeded in obtaining this information, we present it by means of the conjugacy vector of G, ∇_G, which is defined below with the rest of the notation and terminology.

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In the first section, we first develop the basic tools our method relies on and then obtain ∇_G when G belongs to any of the families $\mathcal{G}_1$, $\mathcal{G}_2$ or $\mathcal{G}_3$, i.e., when G_3 is abelian. On the other hand, in Theorem 14 we prove that, if $|G| = p^m$, $c(G) = c$ and $G \in \mathcal{G}_a$ then

$$p^{m-2a} + (p^2-1)((a-1)p^{c-1}+1) \le r(G) \le p^{m-3} + p^c - p^{c-1} + p^2 - 1$$

and we characterize the groups for which each equality holds. We find particularly interesting the characterizations in terms of the commutator subgroups of G.

In the second section we get the conjugacy vector of the p-groups of maximal class of order less than or equal to p^9. These results are a direct consequence of the previous theorems, except for the case when $|G| = p^9$, G_3 is not abelian and $c(G) = c(G/Z(G))$, which requires further reasoning. We also give important information about the different types of groups which appear, such as the degree of commutativity, the maximal normal abelian subgroup or the commutator subgroups.

We would like to underline that some other authors have also considered the conjugacy vector of several types of p-groups. This is the case of M. Hall Jr. and J. Senior (see [2]), and R. James, M.F. Newmann and E.A. O'Brien (see [4]) when determining all 2-groups of order up to 128. In [6], J. Poland obtained ∇_G for the p-groups of maximal class with $r(G) = n(p^2-1)+p^e$, i.e., with minimum conjugacy class number (here, $|G| = p^{2n+e}$ with $e \in \{0,1\}$). We completed Poland's work in [8] by giving ∇_G for the next smallest possible value of $r(G)$, that is, for $r(G) = n(p^2-1)+p^e+(p^2-1)(p-1)$.

Definitions and notation

Throughout this paper, G will represent a p-group of maximal class of order p^m ($m \ge 4$). Let $G_i = [G, \overset{i}{\cdots}, G]$ for $i \ge 2$. Then $G_{m-1} \ne 1$ and $G_i = 1$ for $i \ge m$. If we set $G_0 = G$ and define G_1 by the condition $G_1/G_4 = C_{G/G_4}(G_2/G_4)$, it follows that $|G_i : G_{i+1}| = p$ for $i = 0, \ldots, m-1$. The degree of commutativity of G, $c = c(G)$, is defined as

$$c(G) = \max\{k \le m-2 \mid [G_i, G_j] \le G_{i+j+k} \text{ for all } i, j \ge 1\}.$$

Following N. Blackburn (cf. [1]), we take a couple of elements $s \in G - (G_1 \cup C_G(G_{m-2}))$ and $s_1 \in G_1 - G_2$, and define recursively $s_i = [s_{i-1}, s] \in G_i - G_{i+1}$

for $i = 2, \ldots, m-1$. For $i + j \le m - c - 1$, let $\alpha(i, j) \in \mathbb{F}_p$ be determined by the congruence

$$[s_i, s_j] \equiv s_{i+j+c}^{\alpha(i,j)} \pmod{G_{i+j+c+1}}.$$

It is clear that $\alpha(i, i) = 0$ and $\alpha(i, j) = -\alpha(j, i)$ whenever defined. Also, we have $\alpha(i, j) = \alpha(i + 1, j) + \alpha(i, j + 1)$ for $i + j \le m - c - 2$ and Shepherd's product formula

$$\alpha(i,j)\alpha(i+j+c,k) + \alpha(j,k)\alpha(j+k+c,i) + \alpha(k,i)\alpha(k+i+c,j) = 0$$

for $i + j + k \le m - 2c - 1$ (cf. [7], Lemma 1.6). Another important property is the periodicity $\alpha(i, j) = \alpha(i, j + p - 1)$ for $i + j \le m - c - p$. It follows that there exists at least one $j \in \{2, \ldots, p-1\}$ with $\alpha(1, j) \ne 0$.

We will use the notation $\overline{G} = G/Z(G)$, and the letter H will stand for any maximal subgroup of G, apart from G_1 and $C_G(G_{m-2})$. From [1], Lemma 3.1, we have that H is always a p-group of maximal class. Furthermore, $H_i = G_{i+1}$ for every $i \ge 1$.

As in [11], we define the family $\mathcal{F}$ of the p-groups of maximal class for which $c(G) \ne c(\overline{G})$. Also, for $a \ge 1$, we denote by $\mathcal{G}_a$ the family of all p-groups of maximal class whose largest abelian normal subgroup is G_a. Clearly, $G \in \mathcal{G}_a$ implies $H \in \mathcal{G}_{a-1}$ for $a \ge 2$. In general, we have $c \le m - 2a$ and the equality holds for $G \in \mathcal{F}$ (cf. [9], Lemma 1.6). From the definition of the degree of commutativity, $[G_1, G_i] \le G_{i+c+1}$ always holds. If $c \le m - 4$ and $[G_1, G_i] \le G_{i+c+2}$ for some $i \in \{3, \ldots, m-c-2\}$, we say that G has a jump at G_i. From [9], Theorem 2.4, a group $G \in \mathcal{G}_3$ has at most one jump.

For any subset S of G, we denote by $r_G(S)$ the number of G-conjugacy classes which intersect S, that is,

$$r_G(S) = |\{Cl_G(g) \mid g \in G, Cl_G(g) \cap S \ne \emptyset\}|.$$

In particular, $r(G) = r_G(G)$ stands for the number of conjugacy classes of G. Taking into account Example 1 of [12], we have $r_G(gG_{m-1}) = |C_G(g)|/|C_{\overline{G}}(\bar{g})|$ and, consequently, $r_G(gG_{m-1}) = 1$ or p for any $g \in G$. We note that, if $H \le G$ and $N \subseteq H$ is a normal set of G, then $|G : H| r_G(N) \ge r_H(N)$.

For each normal set N of G and $2 \le i \le m$, let

$$a_i(N, G) = |\{Cl_G(g) \subseteq N \mid |C_G(g)| = p^i\}|.$$

Calling M the set of the elements of N for which $|C_G(g)| = p^i$, we have

$$a_i(N,G) = a_i(M,G) = r_G(M) = \frac{1}{p^{m-i}}|\{g \in N|\ |C_G(g)| = p^i\}|.$$

Then, the conjugacy vector of N relative to G is defined as

$$\nabla_N^G = (a_m(N,G), a_{m-1}(N,G), \ldots, a_2(N,G)).$$

Clearly, if N is the disjoint union of two normal sets S and T, then $\nabla_N^G = \nabla_S^G + \nabla_T^G$.

In some situations, most of the components of ∇_N^G will be zero. In those cases, in order to simplify the notation, we will only write the $a_i(N,G)$ values which are non-zero and, to know what centralizer order they correspond to, we will put that value into brackets with i as a subscript. For instance, the vector $(p-1,0,0,0,0,0)$ will be simply written as $([p-1]_7)$. Also, we will consider two conjugacy vectors to be equal if they give the same form after dropping zeros, although they may have a different number of components at the beginning. So there will be no contradiction in writing equalities such as $\nabla_N^G = \nabla_{\overline{N}}^{\overline{G}}$, in spite of the first vector having one more component than the second.

The conjugacy vector we obtain by setting $N = G$ is just called the conjugacy vector of G and denoted by ∇_G. Since $\nabla_{G_{m-1}}^G = ([p]_m)$ and $\nabla_{G-G_1}^G = ([p^2-p]_3, [(p-1)^2]_2)$ or $([p^2-p]_2)$, according as $c(G)$ is zero or not (see [1], p. 64 and [11], Corollary 2.11), it will be enough to find $\nabla_{G_1-G_{m-1}}^G$. As $a_2(G_1 - G_{m-1}) = a_m(G_1 - G_{m-1}, G) = 0$, we will usually work with the vector

$$\nabla_G' = (a_{m-1}(G_1 - G_{m-1}, G), \ldots, a_3(G_1 - G_{m-1}, G))$$

rather than with ∇_G. It is clear that, from the knowledge of ∇_G', we can derive as a by-product the number of conjugacy classes of G. In fact,

$$r(G) = \begin{cases} p^2 + \sum_{i=3}^{m-1} a_i(G_1 - G_{m-1}, G), & \text{if } c(G) \geq 1; \\ 2p^2 - 2p + 1 + \sum_{i=3}^{m-1} a_i(G_1 - G_{m-1}, G), & \text{if } c(G) = 0. \end{cases}$$

1. General results

LEMMA 1: *For each $g \in G$, $|C_G(g)| = p^{\mu_g}$, where*

$$\mu_g = |\{i|\ 0 \leq i \leq m-1 \text{ and } C_G(g) \cap (G_i - G_{i+1}) \neq \emptyset\}|.$$

Moreover, if $g \in G_1 - G_{m-2}$, then $C_G(g) = C_{G_1}(g)$.

Proof: Since $C_{G_i}(g)/C_{G_{i+1}}(g)$ is isomorphic to a subgroup of G_i/G_{i+1}, it follows that $|C_{G_i}(g)/C_{G_{i+1}}(g)| = 1$ or p, according as $C_{G_i}(g) - C_{G_{i+1}}(g) = C_G(g) \cap (G_i - G_{i+1})$ is empty or not. Then, the first part of the lemma follows from the factorization

$$|C_G(g)| = |C_{G_0}(g)| = \prod_{i=0}^{m-1} |C_{G_i}(g)/C_{G_{i+1}}(g)|.$$

Suppose now that $g \in G_i - G_{i+1}$ with $1 \le i \le m-3$. Set $\widetilde{G} = G/G_{i+2}$. If $[g,x] = 1$ then $\widetilde{x} \in C_{\widetilde{G}}(\widetilde{g}) = C_{\widetilde{G}}(\widetilde{G}_i) = \widetilde{G}_1$, since $c(\widetilde{G}) \ge 1$. So $x \in G_1$ and $C_G(x) = C_{G_1}(x)$. ∎

In connection with this result we have the following lemma.

LEMMA 2:

(i) *If $i + j \le m - c - 1$ and $[G_i, G_j] = G_{i+j+c}$, then $C_G(g) \cap (G_j - G_{j+1}) = \emptyset$ for every $g \in G_i - G_{i+1}$.*

(ii) *If $1 \le i \le m - c - 2$ and $[G_i, G_{m-c-i-1}] = G_{m-1}$, then $|C_G(g)| = |C_{\overline{G}}(\bar{g})|$ for every $g \in G_i - G_{i+1}$.*

Proof: (i) Let $g \in G_i - G_{i+1}$ and $x \in G_j - G_{j+1}$. We have $G_i = \langle g, G_{i+1}\rangle$ and $G_j = \langle x, G_{j+1}\rangle$. Since $[G_{i+1}, G_j]$ and $[G_i, G_{j+1}]$ are subgroups of $G_{i+j+c+1}$, the equality $[G_i, G_j] = G_{i+j+c}$ yields $[g,x] \in G_{i+j+c} - G_{i+j+c+1}$. In particular, $C_G(g) \cap (G_j - G_{j+1}) = \emptyset$.

(ii) As we have just proved, $[g,x] \in G_{m-1} - \{1\}$ for every $x \in G_{m-c-i-1} - G_{m-c-i}$. Consequently, $r_G(gG_{m-1}) = 1$ and $|C_G(g)| = r_G(gG_{m-1})|C_{\overline{G}}(\bar{g})| = |C_{\overline{G}}(\bar{g})|$. ∎

Our next result will play an important role when applying inductive methods.

LEMMA 3: *Let N be a normal set of G.*

(i) *If $|C_G(g)| = |C_{\overline{G}}(\bar{g})|$ for every $g \in N$, then $\nabla_N^G = \nabla_{\overline{N}}^{\overline{G}}$.*

(ii) *If $N \subseteq H$ and $|C_G(g)| = |C_H(g)|$ for every $g \in N$, then $\nabla_N^G = (1/p)\nabla_N^H$.*

Proof: (i) From $|C_G(g)| = |C_{\overline{G}}(\bar{g})|$ for $g \in N$ we derive that $a_m(N,G) = 0$. On the other hand, for $2 \le i \le m-1$,

$$a_i(N,G) = \frac{1}{p^{m-i}} |\{g \in N \mid |C_G(g)| = p^i\}| = \frac{1}{p^{m-i}} |\{g \in N \mid |C_{\overline{G}}(\bar{g})| = p^i\}|$$
$$= a_i(\overline{N}, \overline{G}),$$

since $r_G(gG_{m-1}) = 1$ for $g \in N$ implies that N is a union of cosets of G_{m-1}. Consequently, $\nabla_N^G = \nabla_{\overline{N}}^{\overline{G}}$.
(ii) In this case, we just observe that

$$a_i(N,G) = \frac{1}{p^{m-i}} \left|\{g \in N \mid |C_G(g)| = p^i\}\right| = \frac{1}{p^{m-i}} \left|\{g \in N \mid |C_H(g)| = p^i\}\right| = \frac{1}{p}\, a_i(N,H)$$

holds for $2 \le i \le m-1$. ∎

THEOREM 4: *If $G \in \mathcal{F}$, then*

$$\nabla'_G = \begin{cases} \nabla'_{\overline{G}} + ([p-1]_{m-1}), & \text{if } c = 0; \\ \nabla'_{\overline{G}} + ([p^c-1]_{m-1}, [-p^{c-1}+1]_{m-2}), & \text{if } c \ge 1. \end{cases}$$

Proof: First of all, we suppose $c \le m-4$ and see that $[G_i, G_{m-c-i-1}] = G_{m-1}$ for $1 \le i \le m-c-2$. In fact, since $c(\overline{G}) \ge c+1$, we have $[\overline{G}_i, \overline{G}_j] \le \overline{G}_{i+j+c+1}$, whence $[G_i, G_j] \le G_{i+j+c+1}$ and $\alpha(i,j) = 0$ for $i+j \le m-c-2$. From the relation $\alpha(i,j) = \alpha(i-1,j) - \alpha(i-1,j+1)$ we derive that all the $\alpha(i,j)$ with $i+j = m-c-1$ are simultaneously zero or non-zero. If they were all zero, then $[G_i, G_j] \le G_{i+j+c+1}$ for all $i, j \ge 1$, contradicting the definition of $c(G)$. Consequently, $\alpha(i,j) \ne 0$ and $[G_i, G_j] = G_{i+j+c} = G_{m-1}$ for $i+j = m-c-1$, as we wanted to prove.

Now, Lemma 2 yields $|C_G(g)| = |C_{\overline{G}}(\bar{g})|$ for all $g \in G_1 - G_{m-c-1}$, and we conclude from Lemma 3 that

$$\nabla^G_{G_1 - G_{m-c-1}} = \nabla^{\overline{G}}_{\overline{G}_1 - \overline{G}_{m-c-1}}. \tag{1}$$

We note that this equality is obviously true when $c = m-2$. So it holds for any $G \in \mathcal{F}$.

We have

$$\nabla'_G = \nabla^G_{G_1 - G_{m-c-1}} + \nabla^G_{G_{m-c-1} - G_{m-1}} = \nabla^G_{G_1 - G_{m-c-1}} + ([p^c - 1]_{m-1}). \tag{2}$$

Also, if $c = 0$ then

$$\nabla'_{\overline{G}} = \nabla^{\overline{G}}_{\overline{G}_1 - \overline{G}_{m-1}} - \nabla^{\overline{G}}_{\overline{G}_{m-2} - \overline{G}_{m-1}} = \nabla^{\overline{G}}_{\overline{G}_1 - \overline{G}_{m-1}} - ([p-1]_{m-1}), \tag{3}$$

and, if $c \ge 1$, taking into account that $\overline{G}_{m-c-1} \le Z(\overline{G}_1)$,

$$\nabla'_{\overline{G}} = \nabla^{\overline{G}}_{\overline{G}_1 - \overline{G}_{m-c-1}} + \nabla^{\overline{G}}_{\overline{G}_{m-c-1} - \overline{G}_{m-2}} = \nabla^{\overline{G}}_{\overline{G}_1 - \overline{G}_{m-c-1}} + ([p^{c-1} - 1]_{m-2}). \tag{4}$$

Now, the theorem follows from (1), (2), (3) and (4). ∎

Remark: In particular, Theorem 4 yields

$$r(G) = \begin{cases} r(\overline{G}) + p^{c(G)-1}(p-1), & \text{if } c(G) \geq 1; \\ r(\overline{G}) + p(p-1), & \text{if } c(G) = 0; \end{cases}$$

for $G \in \mathcal{F}$. These equalities are proved in another way in [11], Theorem 2.10.

THEOREM 5: *Suppose that $c(G) \leq m-4$ and $[G_1, G_j] = G_{j+c+1}$ for all $j = 2, \ldots, m-c-2$. Then,*

$$\nabla'_G = \frac{1}{p}\{\nabla'_H - ([p^c - 1]_{m-2})\} + ([p^c - 1]_{m-1}, [p^{c+1} - p^c]_{c+2}).$$

Proof: First of all, we observe that $c \geq 1$. In fact, if $c = 0$ then $[G_1, G_j] \leq G_{j+2}$ for $2 \leq j \leq m-3$. It follows from our hypotheses that $m = 4$, in contradiction with $c = 0$.

Now, let $g \in G_1 - G_2$. From $[G_1, G_j] = G_{j+c+1}$ and Lemma 2, we get $C_G(g) \cap (G_2 - G_{m-c-1}) = \emptyset$. Since $G_{m-c-1} \leq Z(G_1) \leq C_G(g)$ and $g \in C_G(g) \cap (G_1 - G_2)$, we deduce from Lemma 1 that $|C_G(g)| = p^{c+2}$. Then $r_G(G_1 - G_2) = |G_1 - G_2|/p^{m-c-2} = p^{c+1} - p^c$ and $\nabla^G_{G_1-G_2} = ([p^{c+1} - p^c]_{c+2})$.

Next, take $x \in G_2 - G_{m-c-1}$. Let j be such that $x \in G_j - G_{j+1}$. As $[G_1, G_j] = G_{j+c+1}$, we have $C_G(g) \cap (G_1 - G_2) = \emptyset$. Note that $x \in G_1 - G_{m-2}$, since $c \geq 1$. So $C_G(x) = C_{G_1}(x) = C_{G_2}(x) = C_{H_1}(x) = C_H(x)$. From Lemma 3 we derive that

$$\nabla^G_{G_2 - G_{m-c-1}} = \frac{1}{p}\nabla^H_{G_2 - G_{m-c-1}} = \frac{1}{p}\nabla^H_{H_1 - H_{m-c-2}}.$$

Now, since $c(H) \geq c+1$, we have $H_{m-c-2} \leq Z(H_1)$ and

$$\nabla^H_{H_1 - H_{m-c-2}} = \nabla'_H - \nabla_{H_{m-c-2} - H_{m-2}} = \nabla'_H - ([p^c - 1]_{m-2}),$$

what proves the theorem. ∎

We observe that, according to Lemma 1.10 of [9], the hypotheses in the previous theorem hold whenever $c(H) \geq c+2$.

THEOREM 6: *If $G \in \mathcal{G}_1$, then $\nabla'_G = ([p^{m-2} - 1]_{m-1})$ and $r(G) = p^{m-2} + p^2 - 1$.*

THEOREM 7: *If $G \in \mathcal{G}_2$, then $\nabla'_G = ([p^c - 1]_{m-1}, [p^{m-4} - p^{c-1}]_{m-2}, [p^{c+1} - p^c]_{c+2})$ and $r(G) = p^{m-4} + p^{c+1} - p^{c-1} + p^2 - 1$.*

Proof: From [9], Theorem 2.2, we know that $[G_1, G_j] = G_{j+c+1}$ for $j \geq 2$. On the other hand, $\nabla'_H = ([p^{m-3} - 1]_{m-2})$, since $H \in \mathcal{G}_1$. Now it suffices to apply Theorem 5 to obtain ∇'_G. ∎

THEOREM 8: *Let $G \in \mathcal{G}_3$ satisfy $[G_1, G_j] = G_{j+c+1}$ for $j = 2, \ldots, m-c-2$. Then, setting $c' = c(H)$, we have*

$$\nabla'_G = ([p^c - 1]_{m-1}, [p^{c'-1} - p^{c-1}]_{m-2}, [p^{m-6} - p^{c'-2}]_{m-3}, [p^{c'} - p^{c'-1}]_{c'+2}, [p^{c+1} - p^c]_{c+2})$$

and $r(G) = p^{m-6} + p^{c'} - p^{c'-2} + p^{c+1} - p^{c-1} + p^2 - 1$.

Proof: Immediate from Theorems 5 and 7. ∎

Our next step will be to find the conjugacy vector of a p-group $G \in \mathcal{G}_3$ having a jump. With that purpose, in the following lemmas we analyse some of the conjugacy vectors of the form $\nabla_{G_i - G_{i+1}}$ for such a group. We note that, in this case, Theorem 2.4 of [9] yields $c(H) = c+1$ and $[G_2, G_j] = G_{j+c+2}$ for all $j \geq 3$.

LEMMA 9: *Let $G \in \mathcal{G}_3$ have a jump at G_v. Then $C_G(g) \cap (G_v - G_{v+1}) \neq \emptyset$ for all $g \in G_1 - G_2$ and*

$$\nabla^G_{G_1 - G_2} = \begin{cases} ([p^{c+2} - p^{c+1}]_{c+3}), & \text{if } [G_1, G_1] = G_{c+3} \text{ or } G_{c+4}; \\ ([p^{c+3} - p^{c+2}]_{c+4}), & \text{if } [G_1, G_1] = G_{c+5}. \end{cases}$$

Proof: In this proof we use the results established in [9], Theorem 2.4, about the groups $G \in \mathcal{G}_3$ having a jump.

Let $g \in G_1 - G_2$. Since $[G_1, G_j] = G_{j+c+1}$ for $3 \leq j \leq m-c-2$ and $j \neq v$, we have $C_G(g) \subseteq (G_1 - G_2) \cup (G_2 - G_3) \cup (G_v - G_{v+1}) \cup G_{m-c-1}$. Consequently, $|C_G(g)| \leq p^{c+4}$. On the other hand, if $\widetilde{G} = G/G_{v+c+2}$, then $[\widetilde{G}_1, \widetilde{G}_v] = \widetilde{1}$ and $\langle \widetilde{g} \rangle \widetilde{G}_v \leq C_{\widetilde{G}}(\widetilde{g})$. Hence $|C_G(g)| \geq |C_{\widetilde{G}}(\widetilde{g})| \geq p^{c+3}$.

Suppose $[G_1, G_1] = [G_1, G_2] = G_{c+3}$ or G_{c+4}. Then $[G_1, G_3] = G_{c+4}$ or G_{c+5}, respectively, and $[G_2, G_2] = G_{c+5}$. It follows that $[g, x] \neq 1$ for all $x \in G_2 - G_3$, that is, $C_G(g) \cap (G_2 - G_3) = \emptyset$. So $|C_G(g)| = p^{c+3}$ in this case. If $[G_1, G_1] = G_{c+5}$ then $\widetilde{G}_1 \leq C_{\widetilde{G}}(\widetilde{g})$, whence $|C_G(g)| = p^{c+4}$. ∎

LEMMA 10: *Let $G \in \mathcal{G}_3$ have a jump at G_v. Then, the following assertions hold:*

(i) *If $v = m-c-2$, $\nabla^G_{G_v - G_{v+1}} = ([p^{c+1} - p^c]_{m-1})$.*

(ii) *If $3 \leq v < m-c-2$, $\nabla^G_{G_v - G_{v+1}} = ([p^{c+1} - p^c]_{m-2}, [p^{m-v-3} - p^{m-v-4} - p^c + p^{c-1}]_{m-3})$.*

Proof: (i) This is clear, since $[G_1, G_{m-c-2}] = 1$ yields $C_G(g) = G_1$ for all $g \in G_v - G_{v+1}$.

(ii) Suppose that $3 \le v < m-c-2$. If $y \in G_v - G_{v+1}$, we have $G_3 \le C_G(y)$. In addition, $[G_2, G_v] = G_{v+c+2} \ne 1$ implies $C_G(y) \cap (G_2 - G_3) = \emptyset$. So $C_G(g) \subseteq (G_1 - G_2) \cup G_3$ and $|C_G(g)| = p^{m-3}$ or p^{m-2}. Consider the following two sets:

$$\begin{aligned} A &= \{y \in G_v - G_{v+1} \mid |C_G(y)| = p^{m-2}\} \\ &= \{y \in G_v - G_{v+1} \mid C_G(y) \cap (G_1 - G_2) \ne \emptyset\} \end{aligned}$$

and

$$B = \{(x,y) \in (G_1 - G_2) \times (G_v - G_{v+1}) \mid [x,y] = 1\}.$$

Counting the elements in B in two different ways, we get

$$\begin{aligned} \sum_{x \in G_1 - G_2} |C_G(x) \cap (G_v - G_{v+1})| &= \sum_{y \in G_v - G_{v+1}} |C_G(y) \cap (G_1 - G_2)| \\ &= \sum_{y \in A} |C_G(y) \cap (G_1 - G_2)|. \end{aligned} \tag{5}$$

Since, according to Lemma 9, $C_G(x) \cap (G_v - G_{v+1}) \ne \emptyset$ for all $x \in G_1 - G_2$, we have $|C_G(x) \cap (G_v - G_{v+1})| = |C_{G_v}(x) - C_{G_{v+1}}(x)| = (p-1)|C_{G_{v+1}}(x)| = (p-1)|G_{m-c-1}| = (p-1)p^{c+1}$. Similarly, $|C_G(y) \cap (G_1 - G_2)| = (p-1)|C_{G_2}(y)| = (p-1)|G_3| = (p-1)p^{m-3}$ for all $y \in A$. Hence, (5) yields $|A| = p^{c+3} - p^{c+2}$. Consequently, $r_G(A) = (p^{c+3} - p^{c+2})/p^2 = p^{c+1} - p^c$ and $r_G((G_v - G_{v+1}) - A) = (|G_v - G_{v+1}| - |A|)/p^3 = p^{m-v-3} - p^{m-v-4} - p^c + p^{c-1}$, as required. ∎

LEMMA 11: *Let $G \in \mathcal{G}_3$ have a jump at G_v. Then,*

$$\nabla^G_{G_2 - G_3} = \begin{cases} ([p^{c+1} - p^c]_{c+3}), & \text{if } [G_1, G_1] = G_{c+3} \text{ or } G_{c+4}; \\ ([p^{c+2} - p^{c+1}]_{c+4}), & \text{if } [G_1, G_1] = G_{c+5}. \end{cases}$$

Proof: Since G has a jump, $[G_2, G_j] = G_{j+c+2}$ for $3 \le j \le m-c-3$. So $C_G(g) \subseteq (G_1 - G_2) \cup (G_2 - G_3) \cup G_{m-c-2}$ for every $g \in G_2 - G_3$. As $g \in C_G(g) \cap (G_2 - G_3)$, we derive that $|C_G(g)| = p^{c+3}$ or p^{c+4}, according as $C_G(g) \cap (G_1 - G_2)$ is empty or not.

If $[G_1, G_1] = [G_1, G_2] = G_{c+3}$ or G_{c+4}, arguing as in Lemma 9 we get $C_G(g) \cap (G_1 - G_2) = \emptyset$. Otherwise, if $[G_1, G_1] = G_{c+5}$, setting $\widetilde{G} = G/G_{c+5}$ we have $\widetilde{G}_1 \le C_{\widetilde{G}}(\widetilde{g})$ and $|C_G(g)| = p^{c+4}$. ∎

Note that, if $G \in \mathcal{G}_3$ and $c(G) = 0$, then $G \in \mathcal{F}$ and $m = 6$. Since $\overline{G} \in \mathcal{G}_2$, we obtain ∇'_G by applying Theorem 4. So we can suppose $c(G) \ge 1$ in the next theorem.

THEOREM 12: *Let $G \in \mathcal{G}_3$ have a jump at G_v and suppose that $c(G) \geq 1$. Then, one of the following cases holds:*

(i) *$v = 3$ and $[G_1, G_1] = G_{c+4}$. We have*

$$\nabla'_G = ([p^c - 1]_{m-1}, [p^{c+1} - p^{c-1}]_{m-2}, [p^{m-6} - p^c]_{m-3}, [p^{c+2} - p^c]_{c+3})$$

and $r(G) = p^{m-6} + p^{c+2} + p^{c+1} - p^c - p^{c-1} + p^2 - 1$.

(ii) *$v = 3$ and $[G_1, G_1] = G_{c+5}$. We have*

$$\nabla'_G = ([p^c - 1]_{m-1}, [p^{c+1} - p^{c-1}]_{m-2}, [p^{m-6} - p^c]_{m-3}, [p^{c+3} - p^{c+1}]_{c+4})$$

and $r(G) = p^{m-6} + p^{c+3} - p^{c-1} + p^2 - 1$.

(iii) *$4 \leq v < m - c - 2$. We have*

$$\nabla'_G = ([p^c - 1]_{m-1}, [p^{c+1} - p^{c-1}]_{m-2}, [p^{m-6} - p^c]_{m-3}, [p^{c+2} - p^c]_{c+3})$$

and $r(G) = p^{m-6} + p^{c+2} + p^{c+1} - p^c - p^{c-1} + p^2 - 1$.

(iv) *$v = m - c - 2$. We have*

$$\nabla'_G = ([p^{c+1} - 1]_{m-1}, [p^{m-6} - p^{c-1}]_{m-3}, [p^{c+2} - p^c]_{c+3})$$

and $r(G) = p^{m-6} + p^{c+2} + p^{c+1} - p^c - p^{c-1} + p^2 - 1$.

Proof: We know that $\nabla^G_{G_{m-c-1} - G_{m-1}} = ([p^c - 1]_{m-1})$. So, taking into account the preceding lemmas, it suffices to determine $\nabla^G_{G_i - G_{i+1}}$ for $3 \leq i \leq m - c - 2$ and $i \neq v$. For such an i, we have $[G_1, G_i] = G_{i+c+1}$ and $C_G(g) \cap (G_1 - G_2) = \emptyset$ for any $g \in G_i - G_{i+1}$. If $i = m - c - 2$, it follows that $C_G(g) = G_2$ and $\nabla^G_{G_i - G_{i+1}} = ([p^c - p^{c-1}]_{m-2})$. If $i < m - c - 2$, then $[G_2, G_i] = G_{i+c+2}$ whence $C_G(g) = C_{G_3}(g) = G_3$ and $\nabla^G_{G_i - G_{i+1}} = ([p^{m-i-3} - p^{m-i-4}]_{m-3})$. ∎

Our next goal is to obtain general bounds for the number of conjugacy classes of a p-group of maximal class. We need the next lemma.

LEMMA 13: *Suppose that $c(G) \leq m - 4$. Then, the following formulas hold for $i + j \leq m - c - 1$:*

$$\alpha(i, j) = \sum_{k=0}^{i-1} (-1)^k \binom{i-1}{k} \alpha(1, j + k) \tag{6}$$

and

$$(7)\qquad \alpha(i,j)=\sum_{k=0}^{m-c-i-j-1}\binom{m-c-i-j-1}{k}\alpha(i+k,m-c-i-k-1).$$

Proof: For the first formula, proceed by induction on $i\geq 1$. To prove (7), use induction on $r=m-c-i-j-1$. In both cases, take into account the relation $\alpha(i,j)=\alpha(i+1,j)+\alpha(i,j+1)$. ∎

THEOREM 14: *Let $G\in\mathcal{G}_a$ with $c(G)\geq 1$. Then,*

$$p^{m-2a}+(p^2-1)((a-1)p^{c-1}+1)\leq r(G)\leq p^{m-3}+p^c-p^{c-1}+p^2-1.$$

Proof: We first deduce the lower bound for $r(G)$. We argue by induction on $a\geq 1$. The case $a=1$ is clear from Theorem 6. Suppose $a\geq 2$. Taking into account that $c\leq m-4$, we can decompose $r(G)$ as follows:

$$\begin{aligned} r(G)&=r_G(G-G_1)+r_G(G_1-G_2)+r_G(G_2-G_{m-c-1})+r_G(G_{m-c-1})\\ (8)\qquad &=r_G(G_1-G_2)+r_G(G_2-G_{m-c-1})+p^c+p^2-1.\end{aligned}$$

From $|C_G(g)|\geq|\langle g\rangle G_{m-c-1}|=p^{c+2}$ for all $g\in G_1-G_2$, we deduce $r_G(G_1-G_2)\geq p^{c+1}-p^c$. On the other hand, $r_G(G_2-G_{m-c-1})\geq(1/p)r_H(G_2-G_{m-c-1})=(1/p)r_H(H_1-H_{m-c-2})$ and

$$r(H)=p^c+p^2-1+r_H(H_1-H_{m-c-2}),$$

because $H_{m-c-2}\leq Z(H_1)$. Since $a(H)=a-1$, the lower bound holds for $r(H)$, whence

$$r_G(G_2-G_{m-c-1})\geq p^{m-2a}+(p^2-1)(a-2)p^{c-1}-p^{c-1},$$

bearing in mind that $c(H)\geq c+1$. Now, going back to (8) we obtain the desired bound.

The proof for the upper bound will require quite a different approach. If $c=m-2$ the result is evident. Hence, we can suppose $c\leq m-4$ and (8) holds. It then suffices to show that $r_G(G_1-G_{m-c-1})\leq p^{m-3}-p^{c-1}$.

We define

$$u=\min\{j\geq 2|\ \alpha(1,m-c-j)\neq 0\}.$$

By applying (6) we get $\alpha(i,m-c-u)=\alpha(1,m-c-u)\neq 0$ and $[G_i,G_{m-c-u}]=G_{m-u+i}$ for $1\le i\le u-1$. Hence $C_G(g)\cap(G_1-G_u)=\emptyset$ for all $g\in G_{m-c-u}-G_{m-c-u+1}$. So $C_G(g)=G_u$ and $r_G(G_{m-c-u}-G_{m-c-u+1})=p^c-p^{c-1}$.

Also, since $|C_G(g)|\le p^{m-1}$ for $g\in G_{m-c-u+1}-G_{m-c-1}$,

$$r_G(G_{m-c-u+1}-G_{m-c-1})\le p^{c+u-2}-p^c.$$

Finally, let $g\in G_i-G_{i+1}$ with $1\le i\le m-c-u-1$. If $|C_G(g)|=p^{m-1}$ then $g\in Z(G_1)$ and $G_i\le Z(G_1)$. This yields $G_{m-c-u}\le Z(G_1)$, which is impossible. Consequently, $|C_G(g)|\le p^{m-2}$ for all $g\in G_1-G_{m-c-u}$ and

$$r_G(G_1-G_{m-c-u})\le p^{m-3}-p^{c+u-2}.$$

Now,

$$\begin{aligned} r_G(G_1-G_{m-c-1}) =& r_G(G_1-G_{m-c-u})+r_G(G_{m-c-u}-G_{m-c-u+1})\\ &+r_G(G_{m-c-u+1}-G_{m-c-1})\le p^{m-3}-p^{c-1}, \end{aligned}$$

as required. ∎

In the following two theorems we characterize the groups for which the bounds above are attained. We note that, according to Theorem 6, $r(G)$ equals the lower bound for $G\in\mathcal{G}_1$. So it suffices to examine the case $a(G)\ge 2$.

THEOREM 15: *If $c(G)\ge 1$ and $a(G)\ge 2$, the following assertions are equivalent:*

(i) $r(G)=p^{m-2a}+(p^2-1)((a-1)p^{c-1}+1)$.

(ii) $[G_i,G_j]=G_{i+j+c}$ *for* $1\le i\le a-1$ *and* $i<j\le m-c-i-1$.

(iii) $\nabla'_G=([p^c-1]_{m-1},[p^c-p^{c-1}]_{m-i\ (2\le i\le a-1)},[p^{m-2a}-p^{c-1}]_{m-a},[p^{c+1}-p^c]_{c+j\ (2\le j\le a)})$.

Proof: We use induction on $a\ge 2$. If $a=2$ the result is clear from Theorem 7.

If (i) holds, having a look at the proof of Theorem 14, we easily deduce that $|C_G(g)|=p^{c+2}$ for all $g\in G_1-G_2$, $c(H)=c+1$ and that H attains the lower bound. If $[G_1,G_j]\le G_{j+c+2}$ with $2\le j\le m-c-2$, in $\widetilde{G}=G/G_{j+c+2}$ we have $|C_{\widetilde{G}}(\widetilde{g})|\ge|\langle\widetilde{g}\rangle\widetilde{G}_j|=p^{c+3}$, a contradiction. Hence $[G_1,G_j]=G_{j+c+1}$ for $2\le j\le m-c-2$. On the other hand, since (i) holds for H, the inductive hypothesis yields $[H_i,H_j]=H_{i+j+c(H)}=H_{i+j+c+1}$ for $1\le i\le a-2$ and $i<j\le m-c-i-3$. This amounts to $[G_i,G_j]=G_{i+j+c}$ for $2\le i\le a-1$ and $i<j\le m-c-i-1$. So we have proved that (ii) holds.

If we now suppose that (ii) is true, Theorem 5 yields

$$\nabla'_G = \frac{1}{p}\{\nabla'_H - ([p^c - 1]_{m-2})\} + ([p^c - 1]_{m-1}, [p^{c+1} - p^c]_{c+2}).$$

We also deduce that H satisfies (ii) with $a(H) = a - 1$ and $c(H) = c + 1$. So (iii) holds for H and we obtain the desired value of ∇'_G.

Finally, it is obvious that (iii) implies (i). ∎

THEOREM 16: *If $c(G) \geq 1$, the following assertions are equivalent:*
(i) $r(G) = p^{m-3} + p^c - p^{c-1} + p^2 - 1$.
(ii) $[G_i, G_j] \leq G_{m-1}$ *for* $i + j \leq m - c - 1$.
(iii) $\overline{G} \in \mathcal{G}_1$.

Proof: The equivalence between (ii) and (iii) is obvious. So we proceed to show that (i) and (iii) are equivalent. If $c = m-2$ both of them are true. Suppose then $c \leq m-4$. If $\overline{G} \in \mathcal{G}_1$ then $G \in \mathcal{F}$ and, according to the remark after Theorem 4, $r(G) = r(\overline{G}) + p^{c-1}(p-1) = p^{m-3} + p^c - p^{c-1} + p^2 - 1$.

Conversely, suppose (i) holds. From the proof of Theorem 14, we must have $|C_G(g)| = p^{m-2}$ for any $g \in G_1 - G_2$. It follows that $\alpha(1,j) = 0$ for every $j \neq m - c - u$. In particular, $m - c - u \leq p - 1$, since there is at least one $j \in \{2, \ldots, p-1\}$ such that $\alpha(1,j) \neq 0$. Write $m - c - u = 2i + e$ with $e \in \{0,1\}$. If $u \geq 3$, then (6) yields

$$\begin{aligned} 0 = \alpha(i+1, i+1) &= \sum_{k=0}^{i} (-1)^k \binom{i}{k} \alpha(1, i+k+1) \\ &= (-1)^{i+e-1} \binom{i}{1-e} \alpha(1, m-c-u), \end{aligned}$$

whence $\alpha(1, m-c-u) = 0$, a contradiction. Consequently, $u = 2$ and $\alpha(1,j) = 0$ for $2 \leq j \leq m - c - 3$. We deduce from (6) that $\alpha(i,j) = 0$ and $[G_i, G_j] \leq G_{i+j+c+1}$ whenever $i + j \leq m - c - 2$. Hence $c(\overline{G}) \geq c + 1$ and $G \in \mathcal{F}$. Arguing as in the proof of Theorem 4, we get $|C_{\overline{G}}(\bar{g})| = |C_G(g)| = p^{m-2}$ for all $g \in G_1 - G_2$. So $\overline{G}_1$ is abelian and $\overline{G} \in \mathcal{G}_1$. ∎

2. The conjugacy vector for $|G| \leq p^9$

In this section, we determine the conjugacy vector for the p-groups of maximal class of order $\leq p^9$. If $a(G) \leq 3$ and $c(G) \geq 1$, we obtain ∇'_G directly from

Theorems 6, 7, 8 or 12. If $c(G) = 0$ then $G \in \mathcal{F}$ and we can apply Theorem 4, for which we need to know $\nabla'_{\overline{G}}$. Also, for $|G| = p^9$ and $G \in \mathcal{G}_4$, we will use *ad hoc* arguments which rely on the knowledge of the conjugacy vector of $\overline{G}$. So, in these cases, our method will be recursive. The mere listing of the different conjugacy vectors for every group order would lead us to the following problem: when dealing with a specific group G, what is the correct choice of $\nabla'_{\overline{G}}$ among all the possible values? For this reason, together with ∇'_G, we will provide a series of supplementary invariants such as $c(G)$, $a(G)$, the jumps G may have and the commutator $[G_1, G_1] = G'_1$, which will precise the structural differences among groups with different conjugacy vectors. This will help us to identify $\nabla'_{\overline{G}}$ when trying to find the conjugacy vector of a group of larger order. Besides, we assign each group type a number and, together with the above-mentioned invariants, we give the type numbers of $\overline{G}$ and H in our lists. In this way, searching through the tables below, we are able to determine all the commutator subgroups $[G_i, G_j]$ for any group G of order $\leq p^9$. This knowledge will sometimes be useful when applying recursion.

THEOREM 17: *If G has order less than or equal to p^9, then one of the cases listed in Tables* 1, 2, 3 *and* 4 *below holds.*

Proof: As already mentioned, if $a(G) \leq 3$ and $c(G) \geq 1$, or if $G \in \mathcal{F}$, the result is immediate. We just have to take into account that $2c \geq m - 6$ for $G \in \mathcal{G}_3$ (cf. [10]), whence there do not exist groups belonging to $\mathcal{G}_3$ with $|G| = p^9$ and $c(G) = 1$.

So we only need to study the case when $|G| = p^9$, $G \in \mathcal{G}_4$ and $G \notin \mathcal{F}$. Then we have $c(G) = c(\overline{G}) = 1$, $H \in \mathcal{G}_3$, $c(H) = 2$ and $[G_3, G_4] = G_8$. We note that $p \geq 7$ in this case, according to Theorems 3.13 and 3.14 of [1]. Also, since $G \notin \mathcal{F}$, there exist i, j such that $i + j \leq 6$ and $\alpha(i, j) \neq 0$. It follows that $\alpha(1, 2)$ and $\alpha(2, 3)$ can not both be zero.

Set $\alpha(3, 4) = x \neq 0$, $\alpha(2, 5) = y$ and $\alpha(1, 6) = z$. From relation (7) we get $\alpha(1, 2) = \alpha(1, 3) = z + 3y + 2x$, $\alpha(1, 4) = z + 2y + x$, $\alpha(1, 5) = z + y$ and $\alpha(2, 3) = \alpha(2, 4) = y + x$. By applying Shepherd's product formula to the triple $(1, 2, 3)$ we obtain that

$$(z + 3y + 2x)(x - y) + z(x + y) = 0. \tag{9}$$

If $\alpha(2, 3) = 0$ then $x - y \neq 0$ and we deduce that $\alpha(1, 2) = z + 3y + 2x = 0$, a contradiction. Consequently, $\alpha(2,3) = \alpha(2,4) \neq 0$, whence $[G_2, G_3] = G_6$ and $[G_2, G_4] = G_7$.

Now, we consider four cases:

(i) $\alpha(1,6) = \alpha(2,5) = 0$.

In this case, (9) yields $2x^2 = 0$, which is impossible.

(ii) $\alpha(1,6) \neq 0$, $\alpha(2,5) = 0$.

That is, $[G_1, G_6] = G_8$ and $[G_2, G_5] = 1$. From (9) we have $z = -x$. Thus $[G_1, G_2] = G_4$, $[G_1, G_3] = G_5$ and $[G_1, G_4] = [G_1, G_5] = G_7$. Now, straightforward calculations show that G belongs to group type no. 64.

(iii) $\alpha(1,6) = 0$, $\alpha(2,5) \neq 0$.

This amounts to $[G_1, G_6] = 1$, $[G_2, G_5] = G_8$. It follows from Lemmas 2 and 3 that $\nabla^G_{G_2 - G_6} = \nabla^{\overline{G}}_{\overline{G}_2 - \overline{G}_6} = \nabla'_{\overline{G}} - \nabla^{\overline{G}}_{\overline{G}_1 - \overline{G}_2} - ([p-1]_7)$. Also, $C_G(g) = G_1$ for all $g \in G_6 - G_8$, whence $\nabla^G_{G_6 - G_8} = ([p^2 - 1]_8)$.

According to (9), we have $(3y + 2x)(x - y) = 0$. If $x - y = 0$ then $\alpha(1, j) \neq 0$ for $2 \leq j \leq 5$. Thus $\overline{G}$ corresponds to group type no. 27 and it is easily checked that $\nabla'_G = (p^2 - 1, 0, p - 1, p^2 - 1, p^3 - p, 0)$.

If $3y + 2x = 0$ then $\alpha(1,2) = \alpha(1,3) = 0$, $\alpha(1,4) \neq 0$ and $\alpha(1,5) \neq 0$. Hence $[G_1, G_2] = G_{5+\varepsilon}$ with $\varepsilon = 0$ or 1, $[G_1, G_3] = [G_1, G_4] = G_6$ and $[G_1, G_5] = G_7$. Since $\overline{G} \in \mathcal{G}_3$, Lemma 9 yields $|C_{\overline{G}}(\bar{g})| = p^{4+\varepsilon}$ for all $g \in G_1 - G_2$. Consequently, $|C_G(g)| = p^{4+\varepsilon}$ or $p^{5+\varepsilon}$. Let

$$A = \{g \in G_1 - G_2 \mid |C_G(g)| = p^{4+\varepsilon}\}$$

and

$$B = \{g \in G_1 - G_2 \mid |C_G(g)| = p^{5+\varepsilon}\}.$$

Setting $|A| = \lambda_1$ and $|B| = \lambda_2$, we have that

$$\begin{cases} \lambda_1 + \lambda_2 = |G_1 - G_2| = p^8 - p^7, \\ \dfrac{\lambda_1}{p^{4-\varepsilon}} + \dfrac{\lambda_2}{p^{3-\varepsilon}} = r_{G_1}(G_1 - G_2) = (p-1) r_{G_1}(s_1 G_2). \end{cases}$$

Taking into account the proof of Theorem 3.4 of [9], it follows that $r_{G_1}(s_1 G_2) = p^{3+\varepsilon} + p^3 - p^2$. Thus

$$\lambda_1 = \begin{cases} p^8 - 2p^7 + p^6, & \text{if } \varepsilon = 0; \\ p^8 - p^7 - p^6 + p^5, & \text{if } \varepsilon = 1; \end{cases} \quad \text{and} \quad \lambda_2 = \begin{cases} p^7 - p^6, & \text{if } \varepsilon = 0; \\ p^6 - p^5, & \text{if } \varepsilon = 1. \end{cases}$$

From these values we obtain cases 66 and 67 in Table 4.

(iv) $\alpha(1,6) \neq 0$, $\alpha(2,5) \neq 0$.

Since $[G_1, G_6] = [G_2, G_5] = [G_3, G_4] = G_8$, we have $\nabla^G_{G_1 - G_7} = \nabla^{\overline{G}}_{\overline{G}_1 - \overline{G}_7} = \nabla'_{\overline{G}}$. If either of $\alpha(1,3)$ or $\alpha(1,4)$ equals zero, we derive a contradiction from (9). So we have cases 68 or 69. ∎

In the following tables, for some group types, the entry corresponding to the jumps of G is $G \in \mathcal{F}$. We observe that this implies that all the $j \in \{3, \dots, m - c - 3\}$ are jumps in that case.

TABLE 1. Conjugacy vector for $|G| \leq p^6$

No.	$\|G\|$	$c(G)$	$a(G)$	$\overline{G}$ no.	H no.	Jump(s)	G'_1	∇'_G
1	p^4	2	1	–	–	–	1	$(p^2 - 1)$
2	p^5	3	1	1	1	–	1	$(p^3 - 1, 0)$
3	p^5	1	2	1	1	–	G_4	$(p - 1, p^2 - 1)$
4	p^6	4	1	2	2	–	1	$(p^4 - 1, 0, 0)$
5	p^6	2	2	2	2	–	G_5	$(p^2 - 1, p^3 - p, 0)$
6	p^6	1	2	3	2	–	G_4	$(p - 1, p^2 - 1, p^2 - p)$
7	p^6	0	3	2	3	$G \in \mathcal{F}$	G_5	$(p - 1, p^3 - 1, 0)$
8	p^6	0	3	3	3	$G \in \mathcal{F}$	G_4	$(p - 1, p - 1, p^2 - 1)$

TABLE 2. Conjugacy vector for $|G| = p^7$

No.	$c(G)$	$a(G)$	$\overline{G}$ no.	H no.	Jump(s)	G'_1	∇'_G
9	5	1	4	4	–	1	$(p^5 - 1, 0, 0, 0)$
10	3	2	4	4	–	G_6	$(p^3 - 1, p^4 - p^2, 0, 0)$
11	2	2	5	4	–	G_5	$(p^2 - 1, p^3 - p, p^3 - p^2, 0)$
12	1	2	6	4	–	G_4	$(p - 1, p^3 - 1, 0, p^2 - p)$
13	1	3	6	5	–	G_4	$(p - 1, p - 1, p^2 - 1, p^2 - p)$
14	1	3	5	5	G_3	G_5	$(p - 1, p^2 - 1, p^3 - p, 0)$
15	1	3	4	5	G_3	G_6	$(p - 1, p^4 - 1, 0, 0)$
16	1	3	6	5	G_4	G_4	$(p^2 - 1, 0, p^3 - 1, 0)$

TABLE 3. Conjugacy vector for $|G| = p^8$

No.	$c(G)$	$a(G)$	$\overline{G}$ no.	H no.	Jump(s)	G'_1	∇'_G
17	6	1	9	9	–	1	$(p^6 - 1, 0, 0, 0, 0)$
18	4	2	9	9	–	G_7	$(p^4 - 1, p^5 - p^3, 0, 0, 0)$

TABLE 3 (cont.). Conjugacy vector for $|G| = p^8$

No.	$c(G)$	$a(G)$	$\overline{G}$ no.	H no.	Jump(s)	G'_1	∇'_G
19	3	2	10	9	–	G_6	$(p^3-1, p^4-p^2, p^4-p^3, 0, 0)$
20	2	2	11	9	–	G_5	$(p^2-1, p^4-p, 0, p^3-p^2, 0)$
21	1	2	12	9	–	G_4	$(p-1, p^4-1, 0, 0, p^2-p)$
22	2	3	11	10	–	G_5	$(p^2-1, p^2-p, p^3-p, p^3-p^2, 0)$
23	2	3	10	10	G_3	G_6	$(p^2-1, p^3-p, p^4-p^2, 0, 0)$
24	2	3	9	10	G_3	G_7	$(p^2-1, p^5-p, 0, 0, 0)$
25	2	3	11	10	G_4	G_5	$(p^3-1, 0, p^4-p, 0, 0)$
26	1	3	12	10	–	G_4	$(p-1, p^2-1, p^3-p, 0, p^2-p)$
27	1	3	13	11	–	G_4	$(p-1, p-1, p^2-1, p^2-p, p^2-p)$
28	1	3	14	11	G_3	G_5	$(p-1, p^2-1, p^2-p, p^3-p, 0)$
29	1	3	15	11	G_3	G_6	$(p-1, p^2-1, p^4-p, 0, 0)$
30	1	3	16	11	G_4	G_4	$(p-1, p^2-1, p^2-p, p^3-p, 0)$
31	1	3	13	11	G_5	G_4	$(p^2-1, 0, p^2-1, p^3-p, 0)$
32	0	4	9	15	$G \in \mathcal{F}$	G_7	$(p-1, p^5-1, 0, 0, 0)$
33	0	4	10	15	$G \in \mathcal{F}$	G_6	$(p-1, p^3-1, p^4-p^2, 0, 0)$
34	0	4	11	15	$G \in \mathcal{F}$	G_5	$(p-1, p^2-1, p^3-p, p^3-p^2, 0)$
35	0	4	12	15	$G \in \mathcal{F}$	G_4	$(p-1, p-1, p^3-1, 0, p^2-p)$
36	0	4	13	14	$G \in \mathcal{F}$	G_4	$(p-1, p-1, p-1, p^2-1, p^2-p)$
37	0	4	14	14	$G \in \mathcal{F}$	G_5	$(p-1, p-1, p^2-1, p^3-p, 0)$
38	0	4	15	14	$G \in \mathcal{F}$	G_6	$(p-1, p-1, p^4-1, 0, 0)$
39	0	4	16	14	$G \in \mathcal{F}$	G_4	$(p-1, p^2-1, 0, p^3-1, 0)$

TABLE 4. Conjugacy vector for $|G| = p^9$

No.	$c(G)$	$a(G)$	$\overline{G}$ no.	H no.	Jump(s)	G'_1	∇'_G
40	7	1	17	17	–	1	$(p^7-1, 0, 0, 0, 0, 0)$
41	5	2	17	17	–	G_8	$(p^5-1, p^6-p^4, 0, 0, 0, 0)$
42	4	2	18	17	–	G_7	$(p^4-1, p^5-p^3, p^5-p^4, 0, 0, 0)$
43	3	2	19	17	–	G_6	$(p^3-1, p^5-p^2, 0, p^4-p^3, 0, 0)$
44	2	2	20	17	–	G_5	$(p^2-1, p^5-p, 0, 0, p^3-p^2, 0)$
45	1	2	21	17	–	G_4	$(p-1, p^5-1, 0, 0, 0, p^2-p)$
46	3	3	19	18	–	G_6	$(p^3-1, p^3-p^2, p^4-p^2, p^4-p^3, 0, 0)$
47	3	3	18	18	G_3	G_7	$(p^3-1, p^4-p^2, p^5-p^3, 0, 0, 0)$

TABLE 4 (cont.). Conjugacy vector for $|G| = p^9$

No.	$c(G)$	$a(G)$	$\bar{G}$ no.	H no.	Jump(s)	G_1'	∇_G'
48	3	3	17	18	G_3	G_8	$(p^3-1, p^6-p^2, 0, 0, 0, 0)$
49	3	3	19	18	G_4	G_6	$(p^4-1, 0, p^5-p^2, 0, 0, 0)$
50	2	3	20	18	–	G_5	$(p^2-1, p^3-p, p^4-p^2, 0, p^3-p^2, 0)$
51	2	3	22	19	–	G_5	$(p^2-1, p^2-p, p^3-p, p^3-p^2, p^3-p^2, 0)$
52	2	3	23	19	G_3	G_6	$(p^2-1, p^3-p, p^3-p^2, p^4-p^2, 0, 0)$
53	2	3	24	19	G_3	G_7	$(p^2-1, p^3-p, p^5-p^2, 0, 0, 0)$
54	2	3	25	19	G_4	G_5	$(p^2-1, p^3-p, p^3-p^2, p^4-p^2, 0, 0)$
55	2	3	22	19	G_5	G_5	$(p^3-1, 0, p^3-p, p^4-p^2, 0, 0)$
56	1	4	17	24	$G \in \mathcal{F}$	G_8	$(p-1, p^6-1, 0, 0, 0, 0)$
57	1	4	18	24	$G \in \mathcal{F}$	G_7	$(p-1, p^4-1, p^5-p^3, 0, 0, 0)$
58	1	4	19	24	$G \in \mathcal{F}$	G_6	$(p-1, p^3-1, p^4-p^2, p^4-p^3, 0, 0)$
59	1	4	20	24	$G \in \mathcal{F}$	G_5	$(p-1, p^2-1, p^4-p, 0, p^3-p^2, 0)$
60	1	4	22	23	$G \in \mathcal{F}$	G_5	$(p-1, p^2-1, p^2-p, p^3-p, p^3-p^2, 0)$
61	1	4	23	23	$G \in \mathcal{F}$	G_6	$(p-1, p^2-1, p^3-p, p^4-p^2, 0, 0)$
62	1	4	24	23	$G \in \mathcal{F}$	G_7	$(p-1, p^2-1, p^5-p, 0, 0, 0)$
63	1	4	25	23	$G \in \mathcal{F}$	G_5	$(p-1, p^3-1, 0, p^4-p, 0, 0)$
64	1	4	30	25	G_4	G_4	$(p-1, p^2-1, p^2-p, p^3-p, p^3-p^2, 0)$
65	1	4	27	22	G_6	G_4	$(p^2-1, 0, p-1, p^2-1, p^3-p, 0)$
66	1	4	28	22	G_3, G_6	G_5	$(p^2-1, 0, p^2-1, p^3-p, p^3-p^2, 0)$
67	1	4	29	22	G_3, G_6	G_6	$(p^2-1, 0, p^3-1, p^4-p^2, 0, 0)$
68	1	4	27	22	–	G_4	$(p-1, p-1, p-1, p^2-1, p^2-p, p^2-p)$
69	1	4	31	22	G_5	G_4	$(p-1, p^2-1, 0, p^2-1, p^3-p, 0)$

3. Examples

We end this paper providing two examples which show that the bounds for $r(G)$ established in Theorem 14 are actually attained. Also, we give a general example of a p-group of maximal class of order p^9 from which it can be derived the existence of groups for every type listed in Table 4, by just giving values to the parameters in terms of which the group is presented.

Example 1: Let $p \geq 5$ be a prime. For $5 \leq m \leq p$ and $c \in \{1, \ldots, m-4\} \cup \{m-2\}$, $c \leq p-m+2$, B.A. Panferov constructs in [5] a p-group of maximal class G of order p^m, degree of commutativity c and such that $[G_i, G_j] = G_{i+j+c}$

for $1 \leq i < j$. It follows that this group satisfies condition (ii) of Theorem 15 with $a = [(m-c)/2]$.

Example 2: Let $p \geq 3$ be a prime number and $c \geq 1$ a fixed integer. Then, for every $m \equiv c \pmod 2$ satisfying $c + 4 \leq m \leq c + p + 1$, there exists a p-group of maximal class G such that $|G| = p^m$, $c(G) = c$ and $\overline{G} \in \mathcal{G}_1$.

In fact, writing $m = 2n + c$, we can present this group as $G = \langle s, s_i \mid i \geq 1 \rangle$ subject to the following relations:

(R1) $s_i = 1$ for $i \geq m$.

(R2) $s^p = 1$ and $\prod_{k=0}^{p-1} s_{i+k}^{\binom{p}{k+1}} = 1$ for $i \geq 1$.

(R3) $[s_i, s] = s_{i+1}$ for $i \geq 1$ and $[s_i, s_j] = s_{m-1}^{(-1)^j \binom{n-j-1}{i-n} \binom{n-j-1}{n-j-1}}$ for $1 \leq j < i$.

We note that we work with generalized binomial coefficients, that is, for any $r, s \in \mathbb{Z}$,

$$\binom{r}{s} = \begin{cases} \dfrac{r(r-1)\dots(r-s+1)}{s!}, & \text{if } s \geq 1; \\ 1, & \text{if } s = 0; \\ 0, & \text{if } s < 0. \end{cases}$$

In particular, $\binom{r}{s} = 0$ if and only if $s < 0$ or $0 \leq r < s$. Also, $\binom{r-s}{r-s} = 0$ or 1, according as $r < s$ or $r \geq s$.

Example 3: Suppose $p \geq 11$ is a prime. Let $\alpha, \beta, \gamma, \delta, \varepsilon, \lambda, \mu, \nu$ and τ be arbitrary elements of the field $\mathbb{F}_p$ subject to the condition $\tau(\lambda + 2\alpha) = 3\lambda^2$. Then, the following relations define a p-group of maximal class $G = \langle s, s_1, \dots, s_8 \rangle$ of order p^9:

(R1) $s^p = s_i^p = 1$ for $1 \leq i \leq 8$.

(R2) $[s_i, s] = \begin{cases} s_{i+1}, & \text{if } 1 \leq i \leq 7; \\ 1, & \text{if } i = 8. \end{cases}$

(R3) $[s_2, s_1] = s_4^{\alpha} s_5^{\beta} s_6^{\gamma} s_7^{\delta} s_8^{\varepsilon}$, $[s_3, s_1] = s_5^{\alpha} s_6^{\beta-\lambda} s_7^{\gamma-\mu-\alpha\lambda} s_8^{\delta-\nu+(\alpha+\beta)(\tau-\lambda)-\alpha\mu}$,
$[s_4, s_1] = s_6^{\alpha-\lambda} s_7^{\beta-\mu-2\lambda} s_8^{\gamma-\nu-2\mu+\tau+\alpha(\tau-2\lambda)}$, $[s_5, s_1] = s_7^{\alpha-2\lambda} s_8^{\beta-2\mu-3\lambda+2\tau}$,
$[s_6, s_1] = s_8^{\alpha-3\lambda+\tau}$.

(R4) $[s_3, s_2] = s_6^{\lambda} s_7^{\mu} s_8^{\nu}$, $[s_4, s_2] = s_7^{\lambda} s_8^{\mu-\tau}$, $[s_5, s_2] = s_8^{\lambda-\tau}$.

(R5) $[s_4, s_3] = s_8^{\tau}$.

(R6) The rest of the commutator relations are trivial.

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